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The analysis of change, Newton's law of gravity and association models

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Summary. Newton's law of gravity states that the force between two objects in the universe is equal to the product of the masses of the two objects divided by the square of the distance between the two objects. In the first part of the paper it is shown that a model with a 'law-of-gravity' interpretation applies well to the analysis of longitudinal categorical data where the number of people changing their behaviour or choice from one category to another is a measure of force and the goal is to obtain estimates of mass for the two categories and an estimate of the distance between them. To provide a better description of the data dynamic masses and dynamic positions are introduced. It is shown that this generalized law of gravity is equivalent to Goodman's RC(M) association model. In the second part of the paper the model is generalized to two kinds of three-way data. The first case is when there are multiple two-way tables and in the second case we have change over three points of time. Conditional and partial association models are related to three-way distance models, like the INDSCAL model, and triadic distance models respectively.

Keywords: Categorical data; Euclidean distance; Gravity model; Longitudinal data; Square tables; Triadic distance

1. Introduction

This paper will be concerned with longitudinal categorical data, i.e. repeated measurements on a number of observational units with the same instrument. The main interest in studying longitudinal data is whether change occurred and, if so, what the nature of the change is. We shall confine ourselves to the case of categorical data. Our questions concern qualitative change, i.e. changes in attitude, opinion, behaviour or any other categorical variable. This is typically different from continuous data where it might be possible to describe change in terms of better or worse; for categorical data descriptions are in terms of 'different' or 'the same'.

Once longitudinal categorical data have been collected they can be represented in transition frequency tables, which are contingency tables where each way corresponds to the categories of a variable measured at a specific time point (we adopt the way mode terminology for the tables of Carroll and Arabie (1980)). The number of time points defines the number of ways of the transition frequency table. Having measured a group of people twice on a categorical variable, a square transition frequency table arises. If measurements are obtained at three time points the data can be gathered in a three-way contingency table, and so forth.

An example of such data is obtained from Upton and Sārlvik (1981) who discussed changes in political voting in Sweden. The data are shown in Table 1. There are five political parties: the *Communists* COM; the *Social Democrats* SD; the *Centre Party* C; the *People's Party* P; the

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Table 1. Swedish voting data representing voting changes from 1964 (rows) to 1970 (columns)†

	com	sd	c	p	con
COM	(22)	27	4	1	0
SD	16	(861)	57	30	8
C	4	26	(248)	14	7
P	8	20	61	(201)	11
CON	0	4	31	32	(140)

†From Upton and Särlvik (1981).

Conservatives CON. These are the anglicized names following Upton and Sārlvik (1981). The rows correspond to the political parties in 1964 (in capital letters); the columns to the political parties in 1970 (lower-case letters).

The focus will be on change, i.e. on the off-diagonal entries. The values on the diagonal are within parentheses; for these cells special parameters (which are often called loyalty parameters) will be included in the models to be developed.

The question, once we have such change data, is not *whether* there is association but *what the pattern* of association looks like. We shall propose a model for these data based on Newton's law of gravity, which states that the force between any two objects in the universe is proportional to the masses of the two objects and inversely related to the squared distance between the two objects (Newton's law of gravity will be discussed in more detail in the next section). This deterministic model will be applied to the analysis of change where the objects in the universe are the categories of the variable. The force between two objects is measured by the number of respondents making a transition from one category to another. This number is not accurately measured, however, since a sample is obtained from a population. Therefore, the law of gravity is used as a probabilistic model assuming a multinomial sampling scheme (which is the usual sampling scheme for such data). The force will be modelled by the mass of the two categories and a function of the distance between the two objects.

The remainder of this paper is organized as follows. The next section discusses Newton's law of gravity in more detail. Section 3 describes the analysis of change in terms of Newton's law of gravity and introduces dynamic elements in the law to adapt for different data settings. After introducing the dynamic elements it will be shown that the model is a reparameterization of the RC(M) association model (Goodman, 1979, 1991). The usual identification constraints for this model, however, are not suited to the analysis of change. A new way of identifying the solution will be presented and finally the model will be applied to the data in Table 1. In Section 4 the model will be generalized to the case of multiple two-way tables. The gravity models that are developed are related to conditional association models (Clogg, 1982), but again the usual identification constraints are not suited to the analysis of change. Section 5 treats gravitational models for three time points. These models are related to partial association models (Clogg, 1982). Identification and an application to empirical data will be discussed. We shall conclude with discussion and reflection about the results obtained and show some limitations of the work presented.

2. Newton's law of gravity

One of the major laws of the natural sciences is Newton's law of gravity:

> 'All matter attracts all other matter with a force proportional to the product of their masses and inversely proportional to the square of the distance between them'.

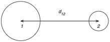

Fig. 1. Newton's law of gravity: the masses of objects 1 and 2 are represented by the area of the circles and d_{12} is the distance between the centres of the two objects

This law can be written in a formula as

$$\mathcal{F}_{ij} \propto \frac{m(i)\,m(j)}{d_{ij}^2}, \tag{1}$$

where $\mathcal{F}_{ij}$ denotes the force between objects i and j, $m(i)$ and $m(j)$ are the masses of the objects i and j, and d_{ij}^2 is the squared distance between the objects. A more general formulation of the law is

$$\mathcal{F}_{ij} \propto \frac{m(i)\,m(j)}{g(d_{ij})}, \tag{2}$$

where $g(\cdot)$ is $g(x) = x^2$, but may also be some other function. A graphical representation is shown in Fig. 1, where the masses are represented by the area of a circle. Newton explained a wide range of previously unrelated phenomena by using this law: the eccentric orbits of comets, the tides and their variations, the precession of the Earth's axis and motion of the Moon as perturbed by the gravity of the Sun. This work made Newton an international leader in scientific research.

In the next section we shall show that the law of gravity applies well to the analysis of social change. Therefore, first some other definitions of the function $g(\cdot)$ are provided and dynamic elements are introduced. The most general model that results is a reparameterization of a well-known model in statistics and social research, the RC(M) association model (Goodman, 1979, 1991). It should be noted, however, that by the time that we arrive at the RC(M) association model many properties of real forces as they are in the natural sciences have been lost. What is maintained is the interpretation in terms of mass and distance, and the analogy with Newton's law of gravity is meant more like a metaphor than reality.

3. The analysis of change

Where the task for Newton was to assess the force of the two objects on each other given their mass and their distance, we deal with the reverse problem. We assume that each object attracts people from other objects with some force. The resultant of these forces is flows of people between objects. These flows can be considered measurements of the attractional forces between objects, and thus (using the law of gravity) are the number of people going from one object to another proportional to the mass of the first object times the mass of the second object and inversely proportional to a function of the distance between the two objects.

In Table 1 it can be seen that there is a large number of people (57) who voted for the Social Democrats in 1964 and for the Centre Party in 1970, so there is a large force between these two categories. Similarly, the force between the Communists and the Conservatives is small (the frequency equals 0). Moreover, the force of one category on another is not equal to the reverse; for example, the force Communists–Social Democrats equals 27 and the force Social Democrats–Communists equals 16.

3.1. A Gaussian link

In Newton's law of gravity the distance is defined by a three-dimensional Euclidean distance, i.e. our universe is three dimensional. For the analysis of change the dimensionality is not known in advance but will be denoted by P. For the data in Table 1, for example, it is likely that the parties are differentiated on the standard 'left–right' dimension that is often found in political systems. Furthermore, there might be another dimension that differentiates the five parties. In Section 5, for example, we find a 'rural–urban' dimension on which the political parties differ. Often the dimensions are interpreted after the solution has been found, on the basis of practical knowledge of the data at hand. The co-ordinates of object i in P-dimensional space are given by the vector $\mathbf{z}(i) = (z_{i1}, \ldots, z_{iP})^{\mathrm{T}}$. The $\mathbf{z}(i)$s will, in turn, be collected in the $I \times P$ matrix $\mathbf{Z} = (\mathbf{z}(1), \mathbf{z}(2), \ldots, \mathbf{z}(I))^{\mathrm{T}}$. The squared Euclidean distance between objects i and j is given by

$$d_{ij}^2(\mathbf{Z}) = \sum_{p=1}^{P} (z_{ip} - z_{jp})^2. \tag{3}$$

Other distance measures might be used as well, e.g. any distance from the Minkowski family (see Borg and Groenen (2005), chapter 17). Where in the law of gravity $g(x) = x^2$ we shall employ $g(x) = \exp(x^2)$, the Gaussian transformation or Gaussian link function (de Rooij and Heiser, 2005; Nosofsky, 1985), which is a monotone function. Again, like for the distance formulation, other transformation functions might be used, but in Section 3.4 it will be shown that this function relates the law of gravity to a well-known model for the statistical analysis of contingency tables.

3.2. Dynamic masses

As discussed above the measured forces are not symmetric, i.e. the force from Communist on Social Democrats is measured to be 27, whereas the reverse force is 16. The law of gravity assumes symmetric forces and the asymmetry is a form of 'error'.

To deal with such asymmetries the model will be generalized in two ways. The first generalization is to make the masses of the objects dependent on time. So, we deal with *dynamic masses*. It is quite natural that masses change in the social sciences, i.e. an object might be popular at one time point and unpopular at another. For example, in brand switching data some brands come into fashion at one moment and go out of fashion another. When an object is popular it has a large mass; when it is unpopular it has a small mass. For our model this means that objects have a mass at each time point, and that mass will be denoted by $m_t(i)$, the mass of object i at time point t. In a graphical representation (like Fig. 1), each object would have two circles.

3.3. Dynamic positions

A second generalization is to make the positions time dependent. So, *dynamic positions* are introduced into the model. An interpretation of a dynamic position is that the content of an object has changed. For example, a political party might change its election programme after it has lost dramatically in the last election or when a loss is in prospect, and thereby change its relative position towards other parties. Each object has a position for each time point which is denoted by $\mathbf{z}_t(i) = (z_{it1}, \ldots, z_{itP})^{\mathrm{T}}$ and the positions of all objects at time point t are gathered in a matrix $\mathbf{Z}_t = (\mathbf{z}_t(1), \mathbf{z}_t(2), \ldots, \mathbf{z}_t(I))^{\mathrm{T}}$. The one-mode Euclidean distance (3) is replaced by a two-mode Euclidean distance:

$$d_{ij}^2(\mathbf{Z}_1; \mathbf{Z}_2) = \sum_{p=1}^{P} (z_{i1p} - z_{j2p})^2. \tag{4}$$

In the graphical representation each object is shown twice: once for each time point.

3.4. Rewriting the model

The model with dynamic masses and dynamic positions is

$$F_{ij} \propto \frac{m_1(i)\, m_2(j)}{\exp\{d_{ij}^2(\mathbf{Z}_1; \mathbf{Z}_2)\}}. \tag{5}$$

This gravity model can be rewritten in a form that is well known in statistics and is often applied in sociological studies; the RC(M) association model (Goodman, 1979, 1991). By back-transforming the parameters of this model, estimates of the masses and co-ordinates of our gravity model are obtained. Furthermore, relationships of this well-known model to other models for contingency tables are well established, and are then also valid for our gravity model. However, the usual graphical displays for the RC(M) association model are susceptible to misinterpretation (for examples see de Rooij and Heiser (2005)), whereas our interpretation is more intuitive. The transformation from gravity to association model is as follows (de Rooij and Heiser, 2005):

$$\begin{aligned} F_{ij} &\propto \frac{m_1(i)\, m_2(j)}{\exp\left\{ \sum_{p=1}^{P} (z_{i1p}^2 + z_{j2p}^2 - 2 z_{i1p} z_{j2p}) \right\}} \\ &\propto \frac{m_1(i)\, m_2(j)}{\exp\left(\sum_{p=1}^{P} z_{i1p}^2 \right) \exp\left(\sum_{p=1}^{P} z_{j2p}^2 \right) \exp\left(\sum_{p=1}^{P} -2 z_{i1p} z_{j2p} \right)}. \end{aligned} \tag{6}$$

Defining $\alpha(i) = m_1(i) / \exp(\Sigma_{p=1}^{P} z_{i1p}^2)$ and $\beta(j) = m_2(j) / \exp(\Sigma_{p=1}^{P} z_{j2p}^2)$, we obtain

$$\begin{aligned} F_{ij} &\propto \frac{\alpha(i)\, \beta(j)}{\exp\left(\sum_{p=1}^{P} -2 z_{i1p} z_{j2p} \right)} \\ &\propto \alpha(i)\, \beta(j) \exp\left(\sum_{p=1}^{P} 2 z_{i1p} z_{j2p} \right) \\ &\propto \alpha(i)\, \beta(j) \exp\left(\sum_{p=1}^{P} \phi_p \mu_{ip} \nu_{jp} \right), \end{aligned} \tag{7}$$

with $z_{i1p} = \phi_p^{1/2} \mu_{ip} / \sqrt{2}$ and $z_{j2p} = \phi_p^{1/2} \nu_{jp} / \sqrt{2}$. The last line in expression (7) is Goodman's (1979, 1991) RC(M) association model. In summary, we started with (an adaptation of) Newton's law of gravity, introduced dynamic elements and ended up with this well-known model. The RC(M) association model is a reduced rank model for the association which equals the saturated model when the dimensionality equals $I-1$ and which equals the (quasi-) independence model when the dimensionality is 0. The model with stable positions is the *homogeneous* RC(M) association model and imposes a symmetry restriction on the association, and thus is a special case of the quasi-symmetry model (Caussinus, 1965). The model with stable masses and positions is a special case of the symmetry model.

Since our focus is on the off-diagonal entries we need parameters for the diagonal entries of the table. These are loyalty parameters for each class, i.e. the model becomes